Trigonometric Lax matrix for the Kowalevski gyrostat on $so(4)$

I.V. Komarov, A.V. Tsiganov

St.Petersburg State University, St.Petersburg, Russia

Abstract

We present trigonometric Lax matrix and classical r-matrix for the Kowalevski gyrostat on $so(4)$ algebra by using auxiliary matrix algebras $so(3,2)$ or $sp(4)$.

In this note we consider the Kowalevski gyrostat with the Hamiltonian

$$H = J_1^2 + J_2^2 + 2J_3^2 + 2\rho J_3 + 2y_1, \qquad \rho \in \mathbb{R}, \tag{1}$$

on a generic orbit of the $so(4)$ Lie algebra with the Poisson brackets

$$\bigl\{J_i\,,J_j\,\bigr\} = \varepsilon_{ijk}J_k\,, \quad \bigl\{J_i\,,y_j\,\bigr\} = \varepsilon_{ijk}y_k\,, \quad \bigl\{y_i\,,y_j\,\bigr\} = \varkappa^2\varepsilon_{ijk}J_k, \tag{2}$$

where ε_{ijk} is the totally skew-symmetric tensor and $\varkappa \in \mathbb{C}$. These brackets are invariant with respect to transformation $y_i \to ay_i$ and $\varkappa \to a\varkappa$, which allows to include scaling parameter a into the Hamiltonian.

Because physical quantities y, J in (1) should be real, $\varkappa^2$ must be real too and algebra (2) is reduced to its two real forms $so(4,\mathbb{R})$ or $so(3,1,\mathbb{R})$ for positive and negative $\varkappa^2$ respectfully and to $e(3)$ for $\varkappa = 0$.

At $\varkappa = 0$ the Lax matrix for the Kowalevski gyrostat on $e(3)$ algebra was found in [1]

$$L_0(\lambda) = \begin{pmatrix} 0 & J_3 & -J_2 & \lambda - \dfrac{y_1}{\lambda} & 0 \\ -J_3 & 0 & J_1 & -\dfrac{y_2}{\lambda} & \lambda \\ J_2 & -J_1 & 0 & -\dfrac{y_3}{\lambda} & 0 \\ \lambda - \dfrac{y_1}{\lambda} & -\dfrac{y_2}{\lambda} & -\dfrac{y_3}{\lambda} & 0 & -J_3 - \rho \\ 0 & \lambda & 0 & J_3 + \rho & 0 \end{pmatrix}. \tag{3}$$

In order to construct this matrix in framework of a general group-theoretical approach the auxiliary algebra $\mathfrak{g} = so(3,2)$ is taken.

At $\varkappa \neq 0$ the Lax matrix for the Kowalevski gyrostat on $so(4)$ is deformation of the matrix $L_0(\lambda)$ [2]

$$L(\lambda) = Y \cdot L_0(\lambda), \qquad Y = \mathrm{diag}\left(1, 1, 1, \frac{\lambda^2}{\lambda^2 - \varkappa^2}, 1\right). \tag{4}$$

The corresponding classical r-matrix has been constructed in [3]. The algebraic nature of the matrix $L(\lambda)$ (4) was mysterious, because diagonal matrix Y does not belong to the auxiliary $so(3,2)$ algebra.

In this note we present simple trigonometric Lax matrix and the corresponding trigonometric r-matrix on $so(3,2)$ or $sp(4)$ algebras. Similar to [1] in order to get the Lax matrices and the r-matrix for the Kowalewski gyrostat on $so(4)$ we use the the auxiliary Lie algebra $\mathfrak{g} = so(3,2)$ in fundamental representation.

Let us describe this auxiliary space by using one antisymmetric matrix and three symmetric matrices

$$S_4 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&-1\\ 0&0&0&1&0 \end{pmatrix}, \qquad Z_1 = \begin{pmatrix} 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \qquad Z_2 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \qquad Z_3 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&0&0&0&0 \end{pmatrix},$$

which are the generators of the $so(3,2)$ algebra. There are three other symmetric matrices

$$H_i = [Z_i, S_4] \equiv Z_i S_4 - S_4 Z_i, \qquad i = 1, 2, 3.$$

and three other antisymmetric matrices

$$S_1 = [Z_2, Z_3], \qquad S_2 = [Z_3, Z_1], \qquad S_3 = [Z_1, Z_2].$$

Four matrices S_k form maximal compact subalgebra $\mathfrak{f} = so(3) \oplus so(2)$ of $so(3,2)$, whereas six matrices Z_i and H_i belong to the complimentary subspace $\mathfrak{p}$ in the Cartan decomposition $\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$. The corresponding Cartan involution is given by $\sigma : \quad X = -X^T$, where $X \in so(3,2)$.

After similarity transformation $L(\lambda) \to Y^{-1/2} L(\lambda) Y^{1/2}$ of the Lax $L(\lambda)$ (4) and change of the spectral parameter $\lambda = \varkappa / \sin\phi$ one gets a trigonometric Lax matrix on the auxiliary $so(3,2)$ algebra

$$L = \frac{\varkappa}{\sin\phi}\Big(Z_1 + \cos\phi H_2\Big) + \sum_{i=1}^{3}\big(\cos\phi\, J_i\, S_i - \varkappa^{-1}\sin\phi\, y_i\, Z_i\big) + (J_3 + \rho) S_4 \tag{5}$$

or

$$
L=\begin{pmatrix}
0 & \cos\phi\, J_3 & -\cos\phi\, J_2 & \dfrac{\varkappa}{\sin\phi}-\dfrac{\sin\phi}{\varkappa}\,y_1 & 0\\[2ex]
-\cos\phi\, J_3 & 0 & \cos\phi\, J_1 & -\dfrac{\sin\phi}{\varkappa}\,y_2 & \dfrac{\varkappa\cos\phi}{\sin\phi}\\[2ex]
\cos\phi\, J_2 & -\cos\phi\, J_1 & 0 & -\dfrac{\sin\phi}{\varkappa}\,y_3 & 0\\[2ex]
\dfrac{\varkappa}{\sin\phi}-\dfrac{\sin\phi}{\varkappa}\,y_1 & -\dfrac{\sin\phi}{\varkappa}\,y_2 & -\dfrac{\sin\phi}{\varkappa}\,y_3 & 0 & -J_3-\rho\\[2ex]
0 & \dfrac{\varkappa\cos\phi}{\sin\phi} & 0 & J_3+\rho & 0
\end{pmatrix}. \tag{6}
$$

In order to consider the real forms $so(4,\mathbb{R})$ or $so(3,1,\mathbb{R})$ we have to use trigonometric or hyperbolic functions for positive and negative $\varkappa^2$, respectively.

If we put $\phi=\varkappa\lambda^{-1}$ and take the limit $\varkappa\to 0$ one gets known rational Lax matrix $L_0(\lambda)$ (3) for the Kowalevski gyrostat on $e(3)$ [1]

$$
L_0=\lambda(Z_1+H_2)+\sum_{i=1}^{3}\bigl(J_iS_i-\lambda^{-1}\,x_i\,Z_i\bigr)+(J_3+\rho)S_4\,.
$$

The Lax matrices $L(\phi)$ and $L_0(\lambda)$ are invariant with respect to the following involutions

$$
L(\phi)\to -L^T(-\phi)\qquad\text{and}\qquad L_0(\lambda)\to -L_0^T(-\lambda),
$$

that are compatible with the Cartan involution σ. Trigonometric Lax matrix has also a standard point symmetry $\phi\to\phi+n\pi$, $n\in\mathbb{Z}$.

It is easy to prove that trigonometric Lax matrix $L(\phi)$ (5) satisfies relation

$$
\Bigl\{\overset{1}{L}(\phi),\,\overset{2}{L}(\theta)\Bigr\}=\Bigl[r_{12}(\phi,\theta),\,\overset{1}{L}(\phi)\Bigr]-\Bigl[r_{21}(\phi,\theta),\,\overset{2}{L}(\theta)\Bigr]\,. \tag{7}
$$

with trigonometric r-matrix

$$
\begin{aligned}
r_{12}(\phi,\theta)\;&=\;\frac{\sin\phi\sin\theta}{\cos^2\phi-\cos^2\theta}\sum_{i=1}^{3}\Biggl(\cos\theta\,H_i\otimes H_i+\cos\phi\,Z_i\otimes Z_i\\
&-\;\frac{\sin\phi\cos\theta}{\sin\theta}S_i\otimes S_i\Biggr)+S_3\otimes S_4+\frac{\cos\phi\sin^2\theta}{\cos^2\theta-\cos^2\phi}\,S_4\otimes S_4\,.
\end{aligned} \tag{8}
$$

Here we use the standard tensor notations

$$
\overset{1}{L}(\phi)=L(\phi)\otimes 1,\qquad \overset{2}{L}(\theta)=1\otimes L(\theta),\qquad r_{21}(\phi,\theta)=\Pi\,r_{12}(\theta,\phi)\,\Pi\,,
$$

and Π is a permutation operator $\Pi X\otimes Y = Y\otimes X\Pi$ of auxiliary spaces. Remark that due to the independent on ϕ and θ item $S_3\otimes S_4$ in (8) the inequality $r_{21}(\phi,\theta)\neq -r_{12}(\theta,\phi)$ takes place.

If we put $\phi=\varkappa\lambda^{-1}$, $\theta=\varkappa\mu^{-1}$ and take the limit $\varkappa\to 0$ one gets rational r-matrix for the Kowalevski gyrostat on $e(3)$ algebra [3]

$$\begin{aligned} r_{12}(\lambda,\mu) &= \frac{\lambda\mu}{\lambda^2-\mu^2}\sum_{i=1}^{3}\Big(H_i\otimes H_i + Z_i\otimes Z_i - \frac{\mu}{\lambda}S_i\otimes S_i\Big) \\ &+ S_3\otimes S_4 + \frac{\lambda^2}{\lambda^2-\mu^2}S_4\otimes S_4\,. \end{aligned}$$

The well known isomorphism $so(3,2)$ and $sp(4)$ algebras allows us to consider 4×4 Lax matrix instead of 5×5 matrix (6). The four generators Z_1, Z_2, Z_3 and S_4 may be represented by different 4×4 real or complex matrices. Below we give one of the possible forms of the 4×4 Lax matrix for the Kowalevski gyrostat

$$\begin{aligned} L_4(\phi) &= i\begin{pmatrix} -e^{-i\phi}J_3 & -e^{-i\phi}\varkappa\sin\phi & \cos\phi\ J_- & 0 \\ e^{-i\phi}\varkappa\sin\phi & e^{-i\phi}\ J_3 & 0 & -\cos\phi\ J_+ \\ \cos\phi\ J_+ & 0 & e^{i\phi}\ J_3 & -e^{i\phi}\varkappa\sin\phi \\ 0 & -\cos\phi\ J_- & e^{i\phi}\varkappa\sin\phi & -e^{i\phi}\ J_3 \end{pmatrix} + \qquad (9) \\ &+ \varkappa^{-1}\begin{pmatrix} 0 & y_- & 0 & y_3 \\ -y_+ & 0 & -y_3 & 0 \\ 0 & y_3 & 0 & -y_+ \\ -y_3 & 0 & y_- & 0 \end{pmatrix} + \rho\sin\phi\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Here $J_\pm = J_1 \pm iJ_2$ and $y_\pm = y_1 \pm iy_2$.

There are few Lax matrices obtained for different deformations of known integrable systems from their undeformed counterpart in the form (4) (see [2, 3] and references within). The main question in construction of these matrices by the Ansatz $L = Y\cdot L_0$ (4) is a choice of a proper matrix Y for a given rational matrix $L_0(\lambda)$. In this note we show that this choice is related to transformation of the rational r-matrix to the trigonometric one.

The trigonometric r-matrix (8) is constant and the corresponding Lax matrix $L(\phi)$ (5) does not contain ordering problem in quantum mechanics. Hence equation (7) holds true in quantum case both for Lax matrices (5) and (9).

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